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NOTE. The editors desire that contributors send in good problems for solution. Let us have a great variety of problems for solutions in the various departments. Also send in solutions; we prefer to publish solutions prepared by contributors, rather than publish our own. We have neither time, inclination, nor ability to solve every problem proposed in the MONTHLY. However, if every contributor will give a little time to the problems each month, by united effort, there will be few problems remain unsolved. We have recently republished a number of unsolved problems, and we shall be pleased to have solutions of any of them, or any others that remain unsolved. Ed, F.

## GEOMETRY.

384. Proposed by S. LEFSHETZ, University of Nebraska.

Let  $ABC$  be a triangle,  $O$  a circle tangent to its three sides,  $T$  a variable tangent of  $O$ , which cuts the sides  $BC$ ,  $CA$ ,  $AB$  in  $a$ ,  $b$ ,  $c$ .  $Oa'$ ,  $Ob'$ ,  $Oc'$  the perpendiculars in  $O$  to  $Oa$ ,  $Ob$ ,  $Oc$ , cutting, respectively,  $T$  in points  $a'$ ,  $b'$ ,  $c'$ . Prove that  $Aa'$ ,  $Bb'$ ,  $Cc'$  meet in a point  $t$ , and find the locus of  $t$  when  $T$  varies. Purely geometrical proofs wanted.

Solution by R. P. BAKER, Iowa City, Iowa.

I. By elementary geometry.

*Lemma I.* The second tangent from  $a'$ ,  $a'K$  to the circle is parallel to  $BC$ .

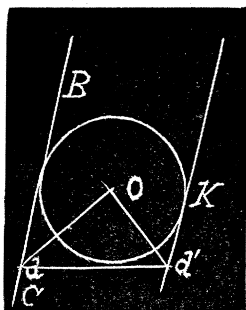


Fig. 1.

For  $a'aB$ ,  $aa'A$  are the doubles of the complementary angles  $a'aO$  and  $aa'O$ , and hence supplementary. So  $a'K$  is parallel to  $aB$ , that is to  $CB$ .

*Lemma II.* If the tangents to a circle from  $P$  meet two parallel tangents in  $Q$ ,  $Q'$ ;  $R$ ,  $R'$ , respectively, and  $O$  is the center; then  $PQ \cdot PR = OP^2 = PQ' \cdot PR'$ .

Let  $M$ ,  $N$  be the points of contact of  $PQ$ ,  $PR$ , and  $S$ ,  $T$  of the parallel tangents.

Then  $OPQ = OPR$ ;  $2OQP = \text{supp. } SOM = \text{supp.}$

$(SOP - MOP) = \text{supp. } SOP + MOP$ .

$2POR = 2POT - TON = 2POT - (POT - NOP) = POT + MOP = \text{supp. } SOP + MOP$ . Therefore,  $OQP = POR$ , and the triangles  $OPQ$  and  $RPO$  are similar.

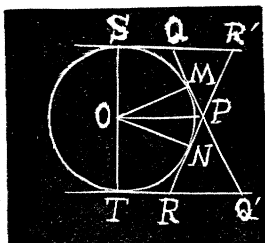


Fig. 2.

Hence  $PQ \cdot PR = OP^2$ , and by the similar triangles  $PRQ'$ ,  $PR'Q$ , each is equal to  $PQ' \cdot PR'$ .

Applying the lemmas to the figure, we have  $ab' \cdot aC = ac' \cdot aB$ , and the triangles  $aBb'$ ,  $aCc'$  are similar, having equal angles at  $a$ , and  $Bb'$ ,  $Cc'$  are parallel. So for  $Aa'$ ,  $Bb'$ .

II. By Brianchon's Theorem.

If the second tangents from  $b'$ ,  $c'$  meet at  $A'$ , we have in  $AB$ ,  $BC$ ,  $CA$ ,  $b'A'$ ,  $bc$ ,  $c'A'$  six tangents to a conic. If the lines are taken in the order written the joins of the cuts of opposite pairs are  $Bb'$ ,  $Cc'$ , and the line at infinity. Hence  $Bb'$ ,  $Cc'$  are parallel, and similarly, the other pairs.

III. Consider the locus of intersections of  $Bb'$  and  $Cc'$ . The pencils at  $B$  and  $C$  are projective, being in 1 : 1 correspondence with the tangents  $T$  by a ruler construction. The intersection is at infinity on  $AB$  when  $T$  coincides with  $AB$ , and at infinity on  $AC$  when  $T$  coincides with  $AC$ . It is on  $BC$  only if  $T$  coincides with  $BC$ . From the latter fact the locus is a straight line, and from the two former the line at infinity, the pencils being in perspective.

#### IV. The generalized problem.

$AB, BC, CA, Ta, OI, OJ$  are six tangents to a conic:  $Ta$  cuts  $AB$  in  $c$ ,  $BC$  in  $a$ ,  $CA$  in  $b$ ;  $OI, OJ$  have chord of contact  $IJ$ :  $Oa'$  is the harmonic conjugate of  $Oa$  with respect to  $OI$  and  $OJ$ :  $a'$  is its cut with  $Ta$ , and so for  $Ob', Oc'$ . Then  $Aa'Bb'Cc'$  are concurrent on  $IJ$ .

Let  $AB$  have contact at  $\gamma$ , and  $OI$  at  $\kappa$ ,  $IJ$  at  $P$ ,  $OJ$  at  $\lambda$ . Let  $PG$  be the second tangent from  $P$ ,  $G$  its point of contact.  $G\gamma$ , the polar of  $P$ , passes through  $O$ , the pole of  $PIJ$ . Let it cut  $IJ$  at  $\pi$ . Then  $PI\pi J$  is harmonic, and so is  $P\kappa\gamma\lambda$ . The latter is the range determined on the tangent  $AB$  by the four tangents  $PG, OI, AB, OJ$ . The range on  $ab$  being equal,  $PG$  cuts this line in  $c'$ .

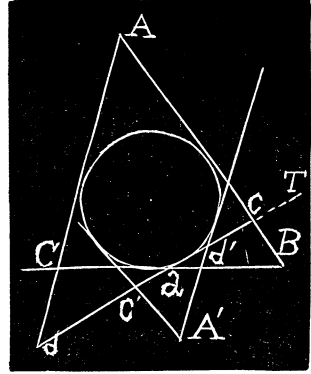


Fig. 3.

This proves Lemma I in general. Brianchon's Theorem is then applied as in II.

#### V. By Analytic Geometry.

Take the equation of the circle as

$$x = \frac{1 - \lambda^2}{1 + \lambda^2}; \quad y = \frac{2\lambda}{1 + \lambda^2}.$$

The tangent at  $\lambda$  is

$$(1 - \lambda^2)x + 2\lambda y = 1 + \lambda^2$$

and intersects the tangent at  $\kappa$  in

$$x = \frac{1 - \kappa\lambda}{1 + \kappa\lambda}; \quad y = \frac{\kappa + \lambda}{1 + \kappa\lambda}.$$

The line through the center perpendicular to the join of this point and the center is:

$$x(1 - \kappa\lambda) + y(\kappa + \lambda) = 0,$$

and cuts the  $\kappa$  tangent in

$$x = \frac{\kappa + \lambda}{\lambda - \kappa}; \quad y = \frac{\kappa\lambda - 1}{\lambda - \kappa}.$$

The line joining this to the cut of tangents at  $(\mu, \nu)$  has the slope

$$\frac{\kappa(\lambda + \mu + \nu + \lambda\mu\nu) - (1 + \lambda\mu + \mu\nu + \nu\lambda)}{2(\kappa + \lambda\mu\nu)}$$

which is symmetrical in  $(\lambda, \mu, \nu)$ . The three lines given by interchanges of  $\lambda, \mu, \nu$  are therefore parallel.

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### CALCULUS.

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312. Proposed by C. N. SCHMALL, New York City.

Given  $y^3 - 3y + x = 0$ , prove by Maclaurin's theorem, that

$$y = \frac{x}{3} + \frac{x^3}{3^4} + \frac{x^5}{3^6} + \text{etc.}$$

I. Solution by H. PRIME.

Put  $u = y^3 - 3y + x = 0$ . Then  $\partial u / \partial x = 1$ ,  $\partial u / \partial y = 3y^2 - 3$ ; hence,  $dy/dx = \frac{1}{3}(1 - y^2) = (1 + y^2 + y^4 + y^6 + \text{etc.})/3$ .

$$\begin{aligned} d^2y/dx^2 &= (2y + 4y^3 + 6y^5 + \text{etc.})(1 + y^2 + y^4 + \text{etc.})/3^2 \\ &= (2y + 6y^3 + 12y^5 + \text{etc.})/3^2. \end{aligned}$$

$$\begin{aligned} d^3y/dx^3 &= (2 + 18y^2 + 60y^4 + \text{etc.})(1 + y^2 + y^4 + \text{etc.})/3^3 \\ &= (2 + 20y^2 + 80y^4 + \text{etc.})/3^3. \end{aligned}$$

$$\begin{aligned} d^4y/dx^4 &= (40y + 320y^3 + \text{etc.})(1 + y^2 + y^4 + \text{etc.})/3^4 \\ &= (40y + 360y^3 + \text{etc.})/3^4. \end{aligned}$$

$$\begin{aligned} d^5y/dx^5 &= (40 + 1080y^2 + \text{etc.})(1 + y^2 + y^4 + \text{etc.})/3^5 \\ &= (40 + 1120y^2 + \text{etc.})/3^5; \text{ etc.} \end{aligned}$$

When  $x=0$ ,  $(y)=0$ . Hence

$(dy/dx) = \frac{1}{3}$ ,  $(d^2y/dx^2) = 0$ ,  $(d^3y/dx^3) = 2/3^3$ ,  $(d^4y/dx^4) = 0$ ,  $(d^5y/dx^5) = 40/3^5$ , etc. By Maclaurin's formula,  $y = x/3 + x^3/3^4 + x^5/3^6 + \text{etc.}$